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# Optimal Interpolating Spaces Preserving Shape

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In this paper we study the existence and characterization of spaces which are images of *minimal-norm* projections that are required to interpolate at given functionals and satisfy additional shape-preserving requirements. We will call such spaces *optimal interpolating spaces preserving shape*. This investigation leads to concrete solutions in classical settings and, as examples, $\Pi_n$ will be determined to be such spaces with regard to certain interpolation and shape-preserving requirements on the projections. Restated, the theory of this paper gives rist to an $n$-dimensional Hahn–Banach extension theorem, where the minimal-norm extension is required to keep invariant a fixed cone. © 1999 Academic Press

## 1. INTRODUCTION AND CHARACTERIZATION OF EXISTENCE

In this paper we study the existence and characterization of spaces which are images of *minimal-norm* projections that are required to interpolate at given functionals and satisfy additional shape-preserving requirements. We will call such spaces *optimal interpolating spaces preserving shape*. This investigation leads to concrete solutions in classical settings and, as examples, $\Pi_n$ will be determined in Section 4 to be such spaces with regard to certain interpolation and shape-preserving requirements on the projections. Restated, the theory of this paper gives rise (see Sections 2 and 3) to an $n$-dimensional Hahn–Banach extension theorem, where the extension is required to keep invariant a fixed cone.



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Let $X$ denote a Banach space and $U$ an $n$-dimensional subspace of the dual space $X^*$. We will use the following notation: an $n$-tuple from $X$ is to be considered a column vector while an $n$-tuple from $X^*$ will be a row vector. Elements of $\mathbf{R}^n$ will be row vectors. Denote by $\mathscr{B}$ the space of bounded linear operators from $X$ to $X$. Given $P \in \mathscr{B}$ with $\ker(P) = U_\perp$, there exists $\vec{u} = (u_1, ..., u_n) \in U^n$ and $\vec{v} = (v_1, ..., v_n)^T \in X^n$ such that the representation $P = \vec{u} \odot \vec{v} = \sum_{i=1}^n u_i \otimes v_i$ is valid, where $Pf = \sum_{i=1}^n \langle f, u_i \rangle v_i$.

DEFINITION 1.1. For a given $n \times n$ nonsingular matrix $A$, $P \in \mathscr{B}$ is said to be an $A$-action operator if $P$ can be written as $P = \sum_{i=1}^n u_i \otimes v_i$ such that $(\langle v_i, u_j \rangle) = A$; i.e., $P\vec{v} = A\vec{v}$.

*Note* 1.1. There is an entire equivalence class of matrices associated with a particular action. That is to say, if $P = \vec{u} \odot \vec{v}$ is an $A$-action operator, then $P$ is also an $MAM^{-1}$-action operator, for any nonsingular matrix $M$, since $P = \vec{u}M \odot M^{-1}\vec{v}$. In the following, it will frequently be advantageous for us to rewrite an operator's representation, as above. To this end, we will resist *fixing* a particular nonsingular matrix $A$ and instead simply refer to a given "action" and use $A$ to denote a representative from the equivalence class.

Let $I$ be the identity operator on an $n$-dimensional subspace $U = [u_1, ..., u_n]$ of a Banach dual space $X^*$ and let $P^* = \sum_{i=1}^n v_i \otimes u_i = \vec{v} \odot \vec{u}$: $X^* \to U$ be linear extension of $I$ to all of $X^*$ of minimal norm, keeping invariant some fixed "proper-shape" cone $S^* \subset X^*$ (i.e. $P^*S^* \subset S^*$). Then (see Theorem 3.2 below), without loss, for some $k$, the vectors $v_{k+1}, ..., v_n$ are determined uniquely by the requirement $P^*S^* \subset S^*$ and $P^*$ is then given by the formula $(\vec{v})_k := (v_1, ..., v_k) = \text{extremal}(M\vec{u})$ for some $k \times n$ matrix $M$. Here "extremal" is defined by use of the $n$-dimensional sphere $\Sigma_n = \{\vec{b}: \|\vec{b} \cdot \vec{u}\|_{X^*} = 1\}$. For example (see Corollary 3.2 below), in the classical case $X = C(T)$, $T$ compact, we have the simple geometric description of $(\vec{v}(t))_k/\lambda$ as a point on $(\Sigma_n)_k$, extremal to $M\vec{u}(t)$, $\forall t \in T$.

In fact we begin our discussion in a more general setting where the action of the operator on the image space is more general than that of the identity operator. We can investigate existence questions in this more general setting. Note of course that, if the action is not the identity action, then the operators do not interpolate at all the $u_i$ (i.e., $\langle Px, u_i \rangle \neq \langle x, u_i \rangle$, $i = 1, ..., n$).

Throughout our discussion, we will want a shape that is given by a subset with particular properties, as specified by the following definition.

DEFINITION 1.2. A *shape* on $X$ is defined via a (non-empty subset $S^* \subset X^*$: $v \in X$ *has shape* (in the sense of $S^*$) provided $\langle v, u \rangle \geqslant 0$ $\forall u \in S^*$. We let $S$ denote the set of all elements with shape and we assume that $S^*$

is such that $S$ contains (at least) $n$ linearly independent elements. We will often refer to $S^*$ as a *shape cone*. (By the term "cone" we mean, as usual, a convex set, closed under nonnegative scalar multiplication.)

*Notation.* Let $\mathscr{A}^U_{S^*}$ denote the set of all shape-preserving $A$-action operators $P$ with kernel $U_\perp$.

THEOREM 1.1 (Characterization A). *$\mathscr{A}^U_{S^*} \neq \emptyset$ if and only if*

$$\exists \vec{v} = (v_1, ..., v_n)^T,\ v_i \in S, \qquad \text{such that,}$$
$$\forall v \in S,\ \vec{\lambda}^v A\vec{v} \in S, \qquad \text{where}$$
$$v_{|_U} = \vec{\lambda}^v \vec{v}_{|_U},\ \vec{\lambda}^v \in \mathbf{R}^n.$$

*Proof.* $(\Rightarrow)$ $P = \vec{u} \odot \vec{v} \in \mathscr{A}^U_{S^*}$ implies, for every $v \in S$,

$$\begin{aligned} Pv &= \langle v, \vec{u} \rangle\, \vec{v} \\ &= \vec{\lambda}^v \,\langle \vec{v}, \vec{u} \rangle\, \vec{v} \\ &= \vec{\lambda}^v A\vec{v} \in S. \end{aligned}$$

$(\Leftarrow)$ Choose $\vec{u} \in U^n$ such that $\langle \vec{v}, \vec{u} \rangle = A$. By use of the fact that $\langle v, \vec{u} \rangle = \vec{\lambda}^v A$ for all $v \in S$, it follows that $P = \vec{u} \odot \vec{v} \in \mathscr{A}^{\mathscr{U}}_{\mathscr{S}^*}$. ∎

COROLLARY 1.1. *Let $U$ be an $n$-dimensional subspace of $X^*$ and let $A$ be an $n \times n$ (action) matrix. If*

$$\exists \vec{v} = (v_1, ..., v_n),\ v_i \in S, \qquad \text{such that,}$$
$$\forall v \in S,\ (\vec{\lambda}^v A)_i \geqslant 0, \qquad i = 1, ..., n, \quad \text{where}$$
$$v_{|_U} = \vec{\lambda}^v \vec{v}_{|_U},\ \vec{\lambda}^v \in \mathbf{R}^n,$$

*then $\mathscr{A}^U_{S^*} \neq \emptyset$.*

*Proof.* If $(\vec{\lambda}^v A)_i \geqslant 0$, $i = 1, ..., n$, then clearly $(\vec{\lambda}^v A) \cdot \vec{v} \in S$, and the result follows from Theorem 1.1. ∎

*Notation.* Let $\delta_t$ denote the functional which evaluates a function at the point $t$ and let $\delta^r_t$ denote the functional which evaluates the $r$th derivative of a function at the point $t$.

EXAMPLE 1.1. Let $X = C[0, 1]$, let $S^*$ be the weak*-closure of the cone generated by $\{\delta_t,\ t \in [0, 1]\}$, and let $U = [\delta_0, \delta_{1/2}, \delta_1] \subset X^*$. Then $S$ is contained in the closure of the cone generated by $\{\chi_{\{t\}}\ \forall t \in [0, 1]\}$. Further consider the basis $\vec{v} = ((1-t)^2, 2t(1-t), t^2)$ of "quadratics" in $X$.

Then for $v=\chi_{\{t\}}$, we have $v_{|_U}=\vec{\lambda}^v\cdot\vec{v}_{|_U}$, whence, for $v=\chi_{\{0\}}, \chi_{\{1/2\}}, \chi_{\{1\}}$, we have $\vec{\lambda}^v=(1, -1/2, 0)$, $(0, 2, 0)$, $(0, -1/2, 1)$, respectively, and $\vec{\lambda}^v=(0, 0, 0)$, $v=\chi_{\{t\}}$, $t\neq 0, 1/2, 1$. (Note that $\chi_{\{t\}}$ itself is not in $S$ but each $v$ which is in $S$ is a linear algebraic (convex) combination of elements $\chi_{\{t\}}\in X^{**}_{|_{\{\delta_t\,|\,t\in[0,1]\}}}$.) Thus Corollary 1.1 will apply to $v$ if it applies to $\chi_{\{t\}}$ for each $t\in[0, 1]$. Now let

$$A=\begin{pmatrix}1 & 1/4 & 0\\ 0 & 1/2 & 0\\ 0 & 1/4 & 1\end{pmatrix}.$$

Then $\vec{\lambda}^v A=(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, for $v=\chi_{\{0\}}, \chi_{\{1/2\}}, \chi_{\{1\}}$, respectively; i.e., all $(\vec{\lambda}^v A)_i$ are always nonnegative on $[0, 1]$. Thus, by Corollary 1.1, $\mathscr{A}^U_{S^*}$ is not empty and the operator $P=\sum_{i=1}^3 u_i\otimes v_i$, where $(\langle v_i, u_j\rangle)=A$ preserves positivity. (Note that $P$ is the classical Bernstein operator onto the quadratics.) Note also that if $A=I$, then $(\vec{\lambda}^v A)_i$ are not always non-negative on $[0, 1]$, reflecting the well-known fact that there does not exist a positivity-preserving projection from $C[0, 1]$ onto the "quadratics" (see [3] for further results regarding the non-existence of more general shape-preserving projections.)

EXAMPLE 1.2. Let $X=C[0, 1]$, let $S^*_{(r)}$ be the cone generated by $\{\delta^r_t|_{C^r[0,1]}: t\in[0, 1]\}$ in $X^*$ (or, alternatively, generated by all $r$th forward differences), $r=0, 1, 2$, and let $U=\{\delta_{t_1}, ..., \delta_{t_n}\}$, for some fixed $0=t_1<t_2<\cdots<t_n=1$. Then the operator $P$, providing the piecewise linear and continuous spline $Px$ interpolating $x$ at the $\{t_i\}_{i=1}^n$, is a projection preserving $r$-convexity of all the orders $r=0, 1, 2$ (corresponding to positivity, monotonicity, and convexity, respectively). Furthermore, since $P$ is clearly of norm one, we see that the space of linear splines described above is an optimal interpolating space preserving positivity, monotonicity, and convexity.

*Note* 1.2. Note that $S$ is always a cone and many subsets $S^*\subset X^*$ may give rise to the same $S$. It is, however, often useful to define $S^*$ "as large as possible" so that $S^*$ and $S$ are "dual" cones, each determining the other (see Lemma 1.1 below). The following definitions accomplish this.

DEFINITION 1.3. The cone $S^*\subset X^*$ is said to be *pointed* if it contains no lines.

*Notation.* Let $B(X)$ denote the unit ball of $X$ and let $\Sigma(X)$ denote the unit sphere ($\{x: \|x\|=1\}$) of $X$.

DEFINITION 1.4. For the cone $S^*$, let $L_{S^*} = \{0\} \cup \{u \in S^* \mid u, -u \in \Sigma(X^*) \cap S^*\}$ and let $E = \bigcap_{u \in L_{S^*}} \ker(u)$.

DEFINITION 1.5. Let $S^* \subset X^*$ be a cone. The shape defined by $S^*$ is said to be *maximally defined* by $S^*$ if both $S^*$ and $S^*_{|_E}$ are weak*-closed subsets of $X^*$ and $E^*$, respectively.

*Note* 1.3. Throughout the following we will want all shapes to be maximally defined. Thus $S^*$ will always denote a cone that maximally defines a shape.

LEMMA 1.1. *$S$ and $S^*$ are "dual" cones in the sense that, if $\langle f, u \rangle \geqslant 0$, $\forall f \in S$, then $u \in S^*$.*

*Proof.* If $S^* = \{0\}$, then there is nothing to prove; thus we assume $S^* \neq \{0\}$. Suppose that $\langle f, u \rangle \geqslant 0$ $\forall f \in S$ but $u \notin S^*$.

We begin by considering the case in which $S^*$ pointed. If $X$ is reflexive we have an immediate contradiction, since $S^*$ being weakly closed and convex can be "separated" from $u$ by a functional $f \in X^{**} = X$ such that $\langle w, f \rangle = \langle f, w \rangle \geqslant 0$ $\forall w \in S^*$ and yet $\langle u, f \rangle = \langle f, u \rangle = -1$ (i.e., $f$ provides a "supporting hyperplane" for $S^*$ separating $S^*$ from $u$); but such an $f$ is in $S$.

In the non-reflexive case the "separating functional" $f$ in $X^{**}$ above is not necessarily in $X$ and therefore not necessarily in $S$ and so the construction of a "separating" hyperplane must be modified as follows. Let $S_1^* = S^* \cap B(X^*)$ and let $S_0^*$ denote the set of extreme points of $S_1^*$ less 0. (Note that $S_1^*$ is the closed convex hull of $S_0^* \cup \{0\}$ by the Krein–Milman theorem.) Let $C = \overline{co}(S_0^*)$, where the closure is with respect to the weak* topology. Note that $C$ is a convex, compact set, not containing the origin. Consider first the case that $u \notin -S^*$ (of course we still suppose that $u \notin S^*$). Then the entire subspace $[u]$ does not intersect $C$ and thus from the convexity and compactness of $C$, it follows that there exists an entire closed hyperplane $H$ containing $[u]$ such that $H \cap C = \varnothing$ (see [6]). Considering $X^*$ with its weak*-topology, let $t \in (X^*/H)^*$ ($t$ not identically zero) and let $q$: $X^* \to X^*/H$ be the natural map. Then $h = t \circ q$ is a (weak*) continous linear functional (with kernel $H$) on $X^*$ and thus $h \in X$ (a continuous linear functional on $X^*$ with its weak*-topology must be in $X$); via scaling we may assume that $\min_{x \in C} \langle x, h \rangle = 1$. Finally we 'shift slightly' the hyperplane so that it strictly separates $C$ from $[u]$. Indeed, let $g \in X$ be such that $\langle u, g \rangle = 1$. If $g \in C^{\perp}$ then $h - g$ strictly separates $C$ and $u$; otherwise let $1/c = \max_{x \in C} \langle x, g \rangle$ whence, for every $x \in C$, we have $\langle x, h - cg \rangle \geqslant 1 - \langle x, cg \rangle \geqslant 0$ and $\langle u, h - cg \rangle = \langle u, -cg \rangle = -c < 0$. In particular we have shown that if $u \notin S^* \cup -S^*$, then $u$ cannot be nonnegative against $S$.

Finally, we consider the case $0 \neq u \in -S^*$. Since $\langle f, u \rangle \geqslant 0 \ \forall f \in S$, we see that $u$ must vanish against $S$. Let $u_1 \in S^*$ be such that $\langle f, u_1 \rangle > 0$ for some $f \in S$ (whose existence is guaranteed by the fact that $S^* \neq \{0\}$). Then the line segment $\lambda u + (1-\lambda) u_1$, $\lambda \in [0, 1]$ does not pass through the origin. Note that every element on this line segment (except for $u$) is positive against $S$. Thus since both $S^*$ and $-S^*$ are closed, there must exist either a (non-zero) element of the line segment that belongs to neither $S^*$ nor $-S^*$ (i.e., there exists an element $\notin S^* \cup -S^*$ that is nonnegative against $S$)—a contradiction to the above; or there is an element on the line segment belonging to $S^* \cap -S^*$ and thus vanishing against $f$. However the only such element is $u$. This would imply $u \in S^*$, which is again a contradiction. We conclude that, in all cases, if $u \notin S^*$ then $u$ cannot be nonnegative against $S$.

In the case that $S^*$ is not pointed, we use arguments very similar to those above but now applied to $S^*_{|_E}$. We first claim that $S^*_{|_E}$ is pointed. Suppose this is not true. Then there exists distinct $\phi_1, \phi_2 \in S^*$ such that $(\phi_1)_{|_E} = w$ while $(\phi_2)_{|_E} = -w$ with $w \neq 0$. Note this implies that $\ker(\phi_1 + \phi_2) \supset E$. We claim $\phi_1 + \phi_2$ does not belong to $\overline{[L_{S^*}]}$, the weak* closure of the linear span of $L_{S^*}$. $\overline{[L_{S^*}]}$ is a closed subspace of the cone $S^*$. Thus if $\phi_1 + \phi_2 \in \overline{[L_{S^*}]}$ then $-(\phi_1 + \phi_2) \in \overline{[L_{S^*}]} \cap S^*$. Whence it would follow that $-\phi_1 \in S^*$ (since $S^*$ is closed under addition) and we would have that $w = 0$, a contradiction. Thus $\phi_1 + \phi_2$ does not belong to $\overline{[L_{S^*}]}$. But this implies that there exists a closed hyperplane strictly separating $\phi_1 + \phi_2$ from $\overline{[L_{S^*}]}$; i.e., there exists $f \in E \subset X$ that does not vanish against $\phi_1 + \phi_2$. This contradiction implies that $S^*_{|_E}$ must be pointed.

In the extreme case that $S^*_{|_E} = \{0\}$, we note that $S^*$ is a (weak*-closed) subspace of $X^*$, and that $S$ is a subspace, when $u \perp S$. Thus if $u \notin S^*$ then, as above, there exists $f \in (S^*)^\perp \subset X$ such that $\langle f, u \rangle = -1$, whence a contradiction follows. Thus we consider the non-trivial (pointed) cone $S^*_{|_E}$ and $u_{|_E}$. An arguement similar to the one given to show $S^*_{|_E}$ pointed shows that $u \notin S^*_{|_E}$. And, arguing as in the case $S^*$ pointed, we see that there exists $f \in E$ such that $\langle x, f \rangle \geqslant 0$ for all $x \in S^*_{|_E}$ and $\langle u, f \rangle < 0$. Since $f \in E \subset X$, we may conclude that $u$ cannot be nonnegative against $S$. ∎

Lemma 1.2. *Let $P \in \mathscr{B}$. Then $PS \subset S \Leftrightarrow P^*S^* \subset S^*$.*

*Proof.* The proof is an immediate consequence of the duality equation $\langle Pf, u \rangle = \langle f, P^*u \rangle$ and Lemma 1.1. ∎

Lemma 1.3. *Let $k$ denote the maximum number of linearly independent elements of $U \cap S^*$ (so $0 \leqslant k \leqslant n$). Let $P = \vec{u} \odot \vec{v} = \sum_{i=1}^n u_i \otimes v_i$, where $[u_1, ..., u_n] = U$ such that $u_i \in S^*$ for $i \leqslant k$, and $v_i \in X$. Then a necessary condition for $P \in \mathscr{A}^{\mathscr{U}}_{\mathscr{S}^*}$ is $v_i \in (S^*)^\perp$ for $i > k$.*

*Proof.* If $P$, as given above, belongs to $\mathscr{A}^{\mathscr{U}}_{\mathscr{S}*}$ then, by Lemma 1.2, $P^*\phi \in U \cap S^* = [u_1, ..., u_k]$. Thus it must be the case that $\langle v_i, \phi \rangle = 0$ for $i > k$. ∎

*Note* 1.4. Lemma 1.3 indicates that the question of existence can be considered as two cases: $k = 0$ and $k = n$. Indeed if $0 < k < n$ then we break the problem into two parts. Setting $U_1 = [u_1, ..., u_k]$ and $U_2 = [u_{k+1}, ..., u_n]$ we study the existence of elements in $\mathscr{A}^{U_1}_{S*}$ and $\mathscr{A}^{U_2}_{S*}$. In this paper, Subsection 1.1 will study the projection action in the case $k = n$ and Section 4 will use the $k = 0$ case.

*Assumption.* We assume that $S$ is *total* over $U$; that is, we assume that $S_{|_U}$ contains $n$ independent elements.

## 1.1. *The Projection Action*

Let $\mathscr{P}^U_{S*}$ denote the set of shape-preserving projections with kernel $U_\perp$. We will show that Corollary 1.1, in the case of projections, results in a simple geometric characterization of $\mathscr{P}^U_{S*}$ (recall that the action matrix for a projection is the identity). This characterization will then lead us to a result concerning uniqueness.

In this section we will assume that the $n$-dimensional subspace $U$ has $n$ linearly independent elements with shape. Note that this implies that the cone $S_{|_U}$ is pointed. Regarding the cone $S$ as a subset of $X^{**}$ one often finds $S$ to possess the following two characteristics: the 'corners' of $S$ form an independent set and the restriction of $S$ to $U$ is closed (in $U^*$). The following definitions summarize these properties.

Definition 1.6. Let $S^{**} \subset X^{**}$ be the weak*-closure of the cone $S \subset X^{**}$. Let $S_1^{**} = S^{**} \cap B(X^{**})$ and let $S_0^{**}$ denote the set of extreme points of $S_1^{**}$ less 0. We will also say that $S^{**}$ is generated by $S_0^{**}$ or by its "edges" $E(S^{**}) = \bigcup_{x \in S_0^{**}} \{\lambda x : \lambda > 0\}$ and write $S^{**} = \overline{cone}(S_0^{**})$ or $S^{**} = \overline{cone}(E(S^{**}))$.

Definition 1.7. We will say that the cone $S^{**}$ is simplicial if $S_0^{**}$ consists of independent elements. Thus if $S^{**}$ has finite dimension $m$ then $S^{**}$ is simplicial is equivalent to $|E(S^{**})| = m$. We will say that $S^*$ is dual-simplicial if $S^{**}$ is simplicial.

Definition 1.8. We will say that $S^*$ is dual-proper (with respect to $U$) if $S_{|_U}$ is closed (in $U^*$).

Theorem 1.2. *Let $S^*$ be dual-simplicial and dual-proper (with respect to $U$). Then $\mathscr{P}^U_{S*} \neq \varnothing$ if and only if the cone $S_{|_U}$ is simplicial.*

*Proof.* ($\Leftarrow$) This direction follows immediately from Corollary 1.1.

($\Rightarrow$) We will show that $|E(S_{|_U})| = n$. Let $P = \mathbf{u} \odot \mathbf{v} \in \mathscr{P}^U_{S^*}$ and note that $(Px)_{|_U} = x_{|_U}$ since $P$ is a projection; and since $PX$ is $n$-dimensional, it follows that $(Px)_{|_U} = (Pw)_{|_U}$ if and only if $Px = Pw$ in $X$. Thus there is a bijection between the $n$-dimensional cones $PS$ and $S_{|_U}$ given by $Px \leftrightarrow x_{|_U}$. This implies that $|E(S_{|_U})| = |E(PS)|$ and we now show $|E(PS)| = n$. Since $S^*$ is proper, it follows again by the Krein–Milman theorem that the compact convex set $S_{|_U} \cap B(X_{|_U})$ is the closed convex hull of its extreme points, and hence (via the identification of $PS$ and $S_{|_U}$) there exists an independent subset $\{Px_1, ..., Px_n\}$ such that each $Px_i \in E(PS)$. (Note that we make the usual identification of a point on the edge with the edge itself.) We will now show that it is impossible for there to be any other edges. Note that for each $i$, $Px_i \in S$ (by Lemma 1.2) and as such may be written as a (possibly infinite) nonnegative combination of elements of $S_0^{**}$; i.e., with $N = U_\perp \cap S_0^{**}$ (note that here we make the usual identification $X \subset X^{**}$), we have

$$Px_i = \int_{S_i} x \, d\mu_i + \int_N x \, d\mu_i, \tag{1}$$

where $\mu_i$ is a positive measure with $\operatorname{supp}(\mu_i) \cap \sim(N) = S_i$. Now taking $P$ of both sides of (1) we find that $Px_i = \int_{S_i} Px \, d\mu_i$, since $P$ is a projection. However, since $Px_i \in E(PS)$, this is only possible if $Px = Px_i$ for all $x \in S_i$. Whence it follows that $x \in S_i$ only if $x_{|_U} = (x_i)_{|_U}$ and thus $S_i \cap S_j = \varnothing$, $i \neq j$. Now, suppose there exists $Px_{n+1} \in E(PS)$ such that $Px_{n+1} \neq Px_i$, $i = 1, ..., n$. Then $Px_{n+1}$ has a representation as in (1), while the $n$-dimensionality of $PX$ implies the existence of constants $c_i$, $i = 1, ..., n$ such that

$$\begin{aligned} \int_{S_{n+1}} x \, d\mu_{n+1} + \int_N x \, d\mu_{n+1} &= Px_{n+1} = c_1 Px_1 + \cdots + c_n Px_n \\ &= \int_{S_1 \cup \cdots \cup S_n} x \, d\mu + \int_N x \, d\mu, \end{aligned} \tag{2}$$

where the (signed) measure $\mu = \sum_{i=1}^n c_i \mu_i$. However, $S_{n+1}$ and $S_1 \cup \cdots \cup S_n$ are disjoint and so (2) contradicts the independence of the set $S_0^{**}$. Thus $|E(PS)| = n$. ∎

Definition 1.9. The shape $S^*$ is said to be strictly dual-proper (with respect to $U$) if $S = S^{**}$ and distinct elements of $S_0^{**}$ do not agree on $U$.

Theorem 1.3. *Let $S^*$ be dual-simplicial and strictly dual-proper. If $\mathscr{P}^U_{S^*} \neq \varnothing$ then $\mathscr{P}^U_{S^*} = \{P\}$.*

*Proof.* Let $E = E(S_{|_V})$. From Theorem 1.2, we have $|E| = n$ and $E = \{x_{1|_U}, ..., x_{n|_U}\}$, where each $x_{i|_V}$ is an edge of $S_{|_U}$. Since $S_{|_U} = cone((S_0^{**})_{|_U})$, it follows that $E \subset (S_0^{**})_{|_U}$, and thus each $x_{i_{|_U}} \in E$ extends uniquely to $x_i \in S_0^{**}$. Then for $P \in \mathscr{P}_{S^*}^U$, we see from the proof of Theorem 1.2 that $Px_i = x_i$ for $i = 1, ..., n$. From here it follows that $P$ is unique. ∎

## 2. CHARACTERIZATION OF MINIMALITY

THEOREM 2.1 (Characterizing Admissible Perturbations). *Let $U$ be an $n$-dimensional subspace of $X^*$. Further, let $S^* \subset X^*$ be a shape cone and $U = U_1 \oplus U_2$, such that $U_1 = U \cap S^\perp = [u_1, ..., u_k]$, where $S \subset X$ is the cone of all elements with shape. Let $U_2 = [u_{k+1}, ..., u_n]$ and let $I$ be the $n \times n$ identity matrix. Suppose that the criteria of Theorem 1.3 are met with respect to $S^*$, $U_2$, and $I'$, where $I'$ denotes the $(n-k) \times (n-k)$ identity matrix induced by $I$ restricted to $U_2$. Then the set $\mathscr{P}_{k,n}$ of all shape-preserving projections with kernel $U_\perp$ is a linear manifold, i.e., $\mathscr{P}_{k,n} = \{P\colon P = P_0 + \Delta\}$, where $P_0$ is a fixed such operator and $\Delta \in \mathscr{D} = \operatorname{span}\{u \otimes \delta\colon \delta \in U_\perp$ and $u \in U_1\}$.*

*Proof.* With $P_1$ as the unique shape-preserving $I'$-action operator respect to $U_2$, we write $P_1 = \sum_{i=n-k+1}^n u_i \otimes v_i$. Now $\{u_1, ..., u_k\} \subset S^\perp$, so we can define $P = P_1 + \sum_{i=n-k+1}^n u_i \otimes v_i$ for any $\{v_1, ..., v_k\} \subset X$ such that $\langle v_i, u_j \rangle = A_{ij}$ for $i, j = 1, ..., n$. Then the set of all such "admissible" $P$ forms the manifold $\mathscr{P} = \{P\colon P = P_0 + \Delta\}$, where $P_0$ is a fixed such operator and $\Delta \in \mathscr{D} = \operatorname{span}\{u \otimes \delta\colon \delta \in U_\perp$ and $u \in U_1\}$. Thus $\mathscr{P}$ is a manifold of shape-preserving projections with respect to $U$. The fact that $P_1$ is unique implies $\mathscr{P}_{k,n} = \mathscr{P}$. ∎

In order to determine minimal-norm operators of $\mathscr{P}_{k,n}$, we will need the following characterization theory for minimal-norm operators from [2] tailored to our situation. We will make use of the form of the projections in $\mathscr{P}_{k,n}$ that was established in Theorem 2.1.

DEFINITION 2.1. If $P$ is a linear operator from $X$ into $X$, then $(x, y) \in \Sigma(X^{**}) \times \Sigma(X^*)$ will be called an extremal pair for $P$ if $\langle P^{**}x, y \rangle = \|P\|$, where $P^{**}\colon X^{**} \to X^{**}$ is the second adjoint extension of $P$ to $X^{**}$ ($\Sigma$ denotes unit sphere).

*Notation.* Let $\mathscr{E}(P)$ be the set of all extremal pairs for $P$. To each $(x, y) \in \mathscr{E}(P)$ associate the rank one operator $x \otimes y$ from $X^*$ to $X^*$ given by $(x \otimes y)(z) = \langle z, x \rangle\ y$ for $z \in X^*$.

THEOREM 2.2 (Characterization of Minimal $P$ in $\mathscr{P}_{k,n}$). *If $\mathscr{P}_{k,n} \neq \varnothing$ then $P$ has minimal norm in $\mathscr{P}_{k,n}$ if and only if the closed convex hull of $\{x \otimes y\}_{(x,y) \in \mathscr{E}(P)}$ contains an operator carrying $U_1$ into $U$.*

*Proof.* Pick $P_0 = \sum_{i=1}^{n} u_i \otimes v_i \in \mathscr{P}_{k,n}$. By Theorem 2.1, the problem of finding a minimal-norm element from $\mathscr{P}_{k,n}$ is equivalent to best approximating, in the operator norm, the fixed operator $P_0 \in \mathscr{P}_{k,k}$ from the space of operators $\mathscr{D}$. Let $K = B(X^{**}) \times B(X^*)$ endowed with the product topology, where $B(\cdot^*)$ denotes the unit ball with its weak* topology. Associate with any operator $Q \in \mathscr{B}$ the bilinear form $\hat{Q} \in C(K)$ via $\hat{Q}(x, y) = \langle Q^{**}x, y \rangle$, and let $\hat{\mathscr{D}} = \{\hat{\Delta} : \Delta \in \mathscr{D}\}$. Then, making use of standard duality theory for $C(K)$, $K$ compact (see e.g., [7, Theorem 1.1, p. 18 and Theorem 1.3, p. 29]), we have that $\hat{P} = \hat{P}_0 - \hat{\Delta}_0$ is of minimal norm if and only if there exists a finite, non-zero (total mass one) signed measure $\hat{\mu}$ supported on the critical set

$$\mathscr{C}(\hat{P}) = \{(x, y) \in \Sigma(X^{**}) \times \Sigma(X^*) : |\hat{P}(x, y)| = \|\hat{P}\|_\infty\}$$

such that $\operatorname{sgn} \hat{\mu}\{(x, y)\} = \operatorname{sgn} \hat{P}(x, y)$ and $\hat{\mu} \in \hat{\mathscr{D}}^\perp$, i.e.,

$$0 = \int_{\mathscr{C}(\hat{P})} \hat{\Delta}\, d\hat{\mu} \qquad \text{for all} \quad \hat{\Delta} \in \hat{\mathscr{D}}.$$

But now, since any $\hat{Q} \in \{\hat{P}\} \cup \hat{\mathscr{D}}$ is a bilinear function, we can replace the signed measure $\hat{\mu}$, supported in $\mathscr{C}(\hat{P})$, by a positive measure $\mu$ supported on $\mathscr{E}(P) \subset \mathscr{C}(\hat{P})$ by noting that

$$\mathscr{C}(\hat{P}) = \{(x, e^{i\theta} y) : (x, y) \in \mathscr{E}(P) \text{ and } \theta \in T\},$$

where $T = [0, 2\pi)$ in the complex case and $T = \{0, \pi\}$ in the real case, and setting

$$\mu\{(x, y)\} = |\hat{\mu}|\ \{(x, e^{i\theta} y) : \theta \in T\}.$$

For then $\operatorname{sgn} \mu\{(x, y)\} = \operatorname{sgn} \hat{P}(x, y) = 1$, for $(x, y) \in \mathscr{E}(P)$ and

$$0 = \int_{\mathscr{E}(P)} \hat{\Delta}\, d\mu \qquad \text{for all} \quad \Delta \in \mathscr{D},$$

since

$$\int_{\mathscr{C}(\hat{P})} \hat{\Delta}\, d\hat{\mu} = \int_{\substack{(x,y) \in \mathscr{E}(P) \\ \theta \in T}} \hat{\Delta}(x, e^{i\theta} y)\, d\hat{\mu}(x, e^{i\theta} y)$$

$$\int_{\substack{(x,y) \in \mathscr{E}(P) \\ \theta \in T}} e^{-i\theta} \hat{\Delta}(x, y)\, e^{i\theta}\, d\, |\hat{\mu}|\, (x, e^{i\theta} y) = \int_{\mathscr{E}(P)} \hat{\Delta}\, d\mu.$$

Hence,

$$0=\int_{\mathscr{E}(P)} \hat{\Delta}\, d\mu=\int_{\mathscr{E}(P)} \langle \Delta^{**}x, y\rangle\, d\mu(x, y)$$

$$=\int_{\mathscr{E}(P)} \langle x, \mu\rangle \langle \delta, y\rangle\, d\mu(x, y)$$

$$=\left\langle \int_{\mathscr{E}(P)} \langle x, u\rangle\, y\, d\mu(x, y), \delta \right\rangle$$

for all $\Delta=u\otimes\delta$ ($\delta\in U^{\perp}$, $u\in U_1$), where, for $z\in X^*$, $\int_{\mathscr{E}(P)} \langle x, z\rangle\, y\, d\mu(x, y)$ is the element $w\in X^*$ defined by $\langle s, w\rangle=\int_{\mathscr{E}(P)} \langle x, \mu\rangle \langle s, y\rangle\, d\mu(x, y)$ for all $s\in X$. $P$ is minimal, therefore, if and only if $\int_{\mathscr{E}(P)} \langle x, u\rangle\, y\, d\mu(x, y)\in (U^{\perp})^{\perp}=U$, i.e., if and only if there exists an operator (from $X^*$ into $X^*$)

$$E_P=\int_{\mathscr{E}(P)} x\otimes y\, d\mu(x, y)\colon U_1\to U. \quad \blacksquare \tag{3}$$

In the examples and discussion below it is helpful to introduce a fixed vector $\vec{u}=(u_1, ..., u_n)$ whose components form a basis for $U$ while the components of $(\vec{u})_k:=(u_1, ..., u_k)$ form a basis for $U_1$. Then the necessary and sufficient condition (3) can be rewritten as a system of $k$ equations

$$\int_{\mathscr{E}(P)} \langle x, (\vec{u})_k\rangle\, y\, d\mu(x, y)=M\vec{u} \qquad \text{for some} \quad k\times n \text{ matrix } M. \tag{4}$$

## 3. GEOMETRY AND EQUATIONS

*Notation.* Denote the underlying real or complex field by $\mathbf{F}$ and introduce the norm on $\mathbf{F}^n$ given by $\|\vec{b}\|=\|\vec{b}\cdot\vec{u}\|_X$. Let $\Sigma_n$ denote the unit sphere of this norm on $\mathbf{F}^n$ and let $(\Sigma_n)_k$ denote the sphere in $\mathbf{F}^k$ obtained by projecting $\Sigma_n$ onto its first $k$ coordinates. The following proposition demonstrates a very useful geometric connection in $\mathbf{F}^n$ between the two components of an extremal pair for any $P\in\mathscr{P}$.

The following proposition is proved in [2] (in dual form). We include the proof for the sake of completeness.

PROPOSITION 3.1 [2]. *For any extremal pair $(x, y)$ of $P=\sum_{i=1}^n u_i\otimes v_i$,*

$$\langle \vec{v}, y\rangle=\|P\|\,\frac{\alpha\, \overline{\langle \{x, \vec{u}\rangle}-\vec{c}^*}{\|\alpha\, \overline{\langle x, \vec{u}\rangle}-\vec{c}^*\|}, \tag{5}$$

*where $\alpha$ is any positive scalar and $\vec{c}^* \in \mathbf{F}^n$ yields* $\min \|\alpha \overline{\langle x, \vec{u}\rangle} - \vec{c}\|$ *subject to* $\vec{c} \cdot \langle x, \vec{u}\rangle = 0$.

*Proof.* Fix $y \in \mathscr{E}(P)$ and let $C = \{\vec{c} \in \mathbf{F}^n : \vec{c} \cdot \langle x, \vec{u}\rangle = 0\}$. Then there exists a positive $\alpha$ $(= |\langle x, \vec{u}\rangle|^2 / \|P\|)$ and $\vec{c}^* \in C$ such that $\langle \vec{v}, y\rangle = \alpha \overline{\langle x, \vec{u}\rangle} - \vec{c}^*$. Hence, $0 = \vec{c} \cdot \langle x, \vec{u}\rangle = \langle \vec{c} \cdot \vec{u}, \ [\mathrm{ext}(\langle \vec{v}, y\rangle \cdot \vec{u})]\rangle = \langle \vec{c} \cdot \vec{u}, \mathrm{ext}(\alpha \overline{\langle x, \vec{u}\rangle} \cdot \vec{v} - \vec{c}^* \cdot \vec{u})\rangle$, for all $\vec{c} \in C$, which implies that $\vec{c}^* \cdot \vec{u}$ is a best approximation to $\alpha \overline{\langle \vec{v}, y\rangle} \cdot \vec{u}$ from $\{\vec{c} \cdot \vec{u} : \vec{c} \in C\}$ with respect to the norm of $X^*$. Hence $\vec{c}^*$ yields the minimum of $\|\alpha \overline{\langle x, \vec{u}\rangle} - \vec{c}\|$. Further, $\|P\| = \overline{\langle \vec{v}, y\rangle} \cdot \langle x, \vec{u}\rangle = \langle (x, \alpha \overline{\langle x, \vec{u}\rangle} - \vec{c}^*) \cdot \vec{u}\rangle = \|(\alpha \overline{\langle x, \vec{u}\rangle} - \vec{c}^*) \cdot \vec{u}\|_X = \|\alpha \overline{\langle x, \vec{u}\rangle} - \vec{c}^*\|$, since $y = \mathrm{ext}(\langle \vec{v}, y\rangle \cdot \vec{u})$. Finally, note that $\alpha$ can be replaced by any other positive quantity by scaling simultaneously the numerator and denominator in (5). ▮

*Note* 3.1. Geometrically, (5) says that $\langle \vec{v}, y\rangle / \|P\|$ is a point of intersection of $\Sigma_n$ and its tangent plane perpendicular (in the ordinary Euclidean sense) to the direction of $\langle x, \vec{u}\rangle$.

THEOREM 3.1 (Geometry of Solution). *If* $P = \sum_{i=1}^n u_i \otimes v_i$ *is minimal in* $\mathscr{P}_{k,n}$, *then*

$$\langle (\vec{v})_k, y\rangle = \|P\| \, \vec{z}(y), \tag{6}$$

*with* $\vec{z}(y)$ *being a point of intersection of* $(\Sigma_n)_k$ *and its tangent plane perpendicular to* $\langle x, (\vec{u})_k\rangle$, *where* $\langle x, (\vec{u})_k\rangle$ *is determined by* (4).

*Proof.* Apply the note following Proposition 3.1 to formula (4). ▮

In many important cases, e.g., in the case $X = C$, formula (4) above reduces to a relatively simple set of equations, from which emerges a remarkably simple geometric solution (Corollary 3.2 below) to the problem of optimal recovery preserving shape. This situation accounts for the relative simplicity of the examples below.

COROLLARY 3.1 (Equations for Minimality). *Let $P$ have minimal norm in $\mathscr{P}_{k,n}$ and suppose that* $\mathscr{E}_2(P) = \{y : (x, y) \in \mathscr{E}(P) \cap \mathrm{supp}(\mu)\}$ *is an independent set. Then, for each* $y \in \mathscr{E}_2(P)$, *let* $y^0 \in (\mathrm{span}\, \mathscr{E}_2(P))^*$ *such that* $\langle y, y^0\rangle = 1$ *and* $\langle z, y^0\rangle = 0$ *for all* $z \neq y$ *in* $\mathscr{E}_2(P)$, *and act on* (4) *with* $y^0$ *to get*

$$\langle x, (\vec{u})_k\rangle \, \mu\{(x, y)\} = M \langle \vec{u}, y^0\rangle. \tag{7}$$

THEOREM 3.2. *Under the hypotheses of Corollary* 3.1, $P = \sum_{i=1}^n u_i \otimes v_i$ *is minimal implies*

$$\langle (\vec{v})_k, y\rangle = \|P\| \, \vec{z}(y),$$

*with* $\vec{z}(y)$ *being a point of intersection of* $(\Sigma_n)_k$ *and its tangent plane perpendicular to* $M\langle \vec{u}, y^0\rangle$.

*Proof.* Apply (7) to (5) and use Note 3.1. ∎

COROLLARY 3.2 (Geometric Interpretation for $C$). *Let* $X^* = C(T)^* \supset U = [\vec{u}]$. *Then* $P = \sum_{i=1}^n u_i \otimes v_i$ *is minimal in* $\mathscr{P}_{k,n}$ *implies*

$$(\vec{v}(t))_k = \|P\| \, \vec{z}(t), \tag{8}$$

*with* $\vec{z}(t)$ *being a point of intersection of* $(\Sigma_n)_k$ *and its tangent plane perpendicular to* $M\vec{u}(t)$ *for some* $k \times n$ *matrix* $M$. (*See Fig.* 1.)

*Proof.* We can take $y_t = \delta_t$ and $y_t^0|_V = \chi_{\{t\}}$ in Theorem 3.2. See [1]. ∎

THEOREM 3.3 ($n$-Dimensional Hahn–Banach Extension Preserving a Cone). *Let* $I$ *be the identity operator on an* $n$*-dimensional subspace* $U = [u_1, ..., u_n]$ *of a Banach dual space* $X^*$ *and let* $P^* = \sum_{i=1}^n v_i \otimes u_i = \vec{v} \odot \vec{u}$: $X^* \to U$ *be a linear extension of* $I$ *to all of* $X^*$ *of minimal norm, keeping invariant some fixed "proper-shape" cone* $S^* \subset X^*$ (*i.e.*, $P^*S^* \subset S^*$). *Then, without loss, for some* $k$, $v_{k+1}, ..., v_n$ *are determined uniquely by the requirement* $P^*S^* \subset S^*$ *and* $P^*$ *is then given by the formula* $(\vec{v})_k := (v_1, ..., v_k) =$ *extremal* $(M\vec{u})$ *for some* $k \times n$ *matrix* $M$. *Here "extremal" is defined by use of the n-dimensional sphere* $\Sigma = \{\vec{b}\colon \|\vec{b} \cdot \vec{u}\|_{X^*} = 1\}$. *E.g., in the classical case* $X = C(T)$, $T$ *compact, we have the simple geometric description of* $(\vec{v}(t))_k/\lambda$ *as a point on* $(\Sigma)_k$, *extremal to* $M\vec{u}(t)$, $\forall t \in T$, *where* $\lambda = \|P\|$.

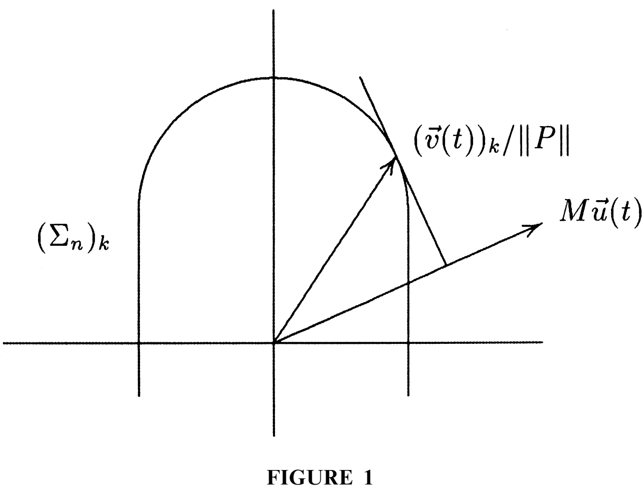


**FIGURE 1**

*Proof.* The theorem is an immediate interpretation of the above theory in conjunction with Theorem 1.1. ∎

## 4. IDENTIFYING $\Pi_n$ AS OPTIMAL INTERPOLATING SPACES PRESERVING SHAPE

THEOREM 4.1. *Let* $(X, \|\cdot\|) = (C^1[0,1], \|\cdot\|)$, *where* $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$. *Let* $U = [u_1, u_2, u_3]$, *where* $(u_1, u_2, u_3) = (\frac{1}{2}(\delta_0 + \delta_1), \ \delta'_0, \delta'_1)$, *let* $A = I$, *and let* $S^*$ *be a shape cone such that* $S = cone(t, t^2)$ (*the cone generated by the functions* $t$ *and* $t^2$); *i.e.,* $\mathscr{A}^U_{S^*}$ *is the set of all projections from* $C^1[0,1]$ *onto* 3-*dimensional subspaces containing* $t$ *and* $t^2$, *interpolating the derivatives at* 0 *and* 1 *and preserving the average of the values at* 0 *and* 1. *Then there is an operator of minimal norm* (3/2) *and with range space* $\Pi_2$, *the subspace of all second-degree algebraic polynomials. This projection has then the explicit form*

$$P = (\tfrac{1}{2}(\delta_0 + \delta_1) - \tfrac{1}{4}(\delta'_0 + \delta'_1)) \otimes 1 + \delta'_0 \otimes t + \tfrac{1}{2}(\delta'_1 - \delta'_0) \otimes t^2. \tag{9}$$

*Proof.* We apply Theorem 2.2 as follows.

First $S$ is clearly total and proper over $U_2 = [\delta'_0, \delta'_1]$ and $U_1 = U \cap S^\perp = [\frac{1}{2}(\delta_0 + \delta_1) - \frac{1}{4}(\delta'_0 + \delta'_1)]$.

Next it is immediate that $P$ is a projection such that $PS \subset S$. Now we calculate that $P$ has norm is 3/2 and determine two appropriate extremal pairs. Since the space $C^1[0,1]$ is normed by $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, we have

$$\|P\| = \sup_{x \in B(X)} \|Px\| = \sup_{x \in B(X)} \max\{\|Px\|_\infty, \|(Px)'\|_\infty\}.$$

But, for $x \in B(X)$,

$$\|(Px)'\|_\infty = \|(1-t)\, x'(0) + t x'(1)\|_\infty \leqslant 1.$$

Thus, when calculating $\|P\|$, we can restrict our attention to $\|Px\|_\infty$. Note that

$$\sup_{x \in \Sigma(X)} |Px(1)| = \sup_{x \in \Sigma(X)} \frac{1}{2}\left(x(1) + \frac{x'(1)}{2} + \frac{x'(0)}{2} + x(0)\right) = \frac{3}{2}.$$

It follows that $\sup_{x \in \Sigma(X)} |Px(0)| = \frac{3}{2}$ as well. Finally, since

$$Px(t) = \frac{x(0)}{2} + \frac{x(1)}{2} - \frac{x'(1)}{4} - \frac{x'(0)}{4} + (x'(0))\,t + \left(\frac{x'(1) - x'(0)}{2}\right) t^2,$$

we have

$$\begin{aligned}|Px(t)| &= \tfrac{1}{2}(x(0)+x(1)+\tfrac{1}{2}(x'(1)(2t^2-1)-x'(0)(2t^2-4t+1))\\ &\leqslant \tfrac{1}{2}(1+1+\tfrac{1}{2}(1+1))=\tfrac{3}{2}\end{aligned}$$

and indeed $\|P\|=\frac{3}{2}$. We next exhibit two extremal pairs of $P$. As we will see, the extremal pairs will be of the form $(w, \delta_0)$ and $(z, \delta_1)$, where $w$ and $z$ are elements of $\Sigma(X^{**})$. Before constructing these pairs, let us briefly consider elements in $X^{**}$. Let $w_n$ be a sequence of functions in $\Sigma(X)$ such that the set

$$M=\left\{f\in X^* \,\middle|\, \lim_{n\to\infty} \langle w_n, f\rangle \text{ exists}\right\}$$

$\neq\{0\}$. $M$ is a subspace of $X^*$. Define on $M$ the linear functional $w$: $M\to\mathbf{R}$ by

$$\langle f, w\rangle = \lim_{n\to\infty} \langle w_n, f\rangle.$$

By the Hahn–Banach extension theorem, extend $w$ to all $X^*$. Of course, we don't know the representation of $w$ off $M$.

With this construction in mind, consider the following family of second-degree polynomials. For each positive integer $n$ define the polynomial $\psi_n(t)=\sum_{i=0}^{2} c_i t^i$ where $c_0=1$ and

$$c_i=(-1)^i \frac{1}{i!\,(2-i)!}\, n^{i-1}$$

for $i=1, 2$. It is easy to check that, for all $n$, we have $\psi_n(0)=1$, $\psi_n'(0)=-1$, $\psi_n(1/n)=1-(1/2n)$ and $\psi_n'(1/n)=-1+(1/n)$. We define now the following sequence:

$$w_n(t)=\begin{cases}\psi_n(t), & t\in\left[0, \dfrac{1}{n}\right]\\[2mm] 1-\dfrac{1}{2n}, & t\in\left[\dfrac{1}{n}, 1-\dfrac{1}{n}\right]\\[2mm] 2\left(1-\dfrac{1}{2n}\right)-\psi_n(1-t), & t\in\left[1-\dfrac{1}{n}, 1\right].\end{cases}$$

Using the definition of $\psi_n(t)$, it is clear that $\{w_n\}\subset X$. Furthermore, one easily checks the following properties of $\{w_n\}$:

$$\|w_n\|=1, \quad w_n(0)=1, \quad w_n'(0)=-1, \quad w_n(1)=1-\frac{1}{n}, \quad w_n'(1)=-1.$$

Note that then

$$w_n(t) = \begin{cases} \frac{n}{2}t^2 - t + 1, & t \in \left[0, \frac{1}{n}\right] \\ 1 - \frac{1}{2n}, & t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \\ \frac{-n}{2}t^2 + (n-1)t - \frac{n^2 - 4n + 2}{2n}, & t \in \left[1 - \frac{1}{n}, 1\right]. \end{cases}$$

Now note that the subspace $M = \{f \in X^* \mid \lim_{n \to \infty} \langle w_n, f \rangle \text{ exists}\}$ contains all point evaluations and derivative point evaluations. Thus, as done above, associate the sequence $\{w_n\}$ with $w \in X^{**}$ and note that in fact $w \in \Sigma(X^{**})$. We claim that $(w, \delta_0)$ is an extremal pair for $P$. Using the definition of the association between $w$ and $\{w_n\}$, we find

$$\begin{aligned} \langle P^{**}w, \delta_0 \rangle = \Big\langle & \lim_{n \to \infty} \left(\tfrac{1}{2}(\langle w_n, \delta_0 \rangle + \langle w_n, \delta_1 \rangle)\right. \\ & \left. - \tfrac{1}{4}(\langle w_n, \delta_1' \rangle + \langle w_n, \delta_0' \rangle)\right), \delta_0 \Big\rangle \Big\langle \lim_{n \to \infty} \langle w_n, \delta_0' \rangle\, t, \delta_0 \Big\rangle \\ & + \Big\langle \tfrac{1}{2} \lim_{n \to \infty} (\langle w_n, \delta_1' \rangle - \langle w_n, \delta_0' \rangle)\, t^2, \delta_0 \Big\rangle \\ = & \lim_{n \to \infty} \left(\tfrac{1}{2}(\langle w_n, \delta_0 \rangle + \langle w_n, \delta_1 \rangle) - \tfrac{1}{4}(\langle w_n, \delta_1' \rangle + \langle w_n, \delta_0' \rangle)\right) \\ = & \tfrac{3}{2}. \end{aligned}$$

With similar motivation, define the sequence $z_n(t) = w_n(1-t)$ in $X$. Thus we have

$$z_n(t) = \begin{cases} -\frac{n}{2}t^2 + t + \frac{n-1}{n}, & t \in \left[0, \frac{1}{n}\right] \\ 1 - \frac{1}{2n}, & t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right] \\ \frac{n}{2}t^2 + (1-n)t + \frac{n}{2}, & t \in \left[1 - \frac{1}{n}, 1\right]. \end{cases}$$

Let $z \in \Sigma(X^{**})$ be associated with $\{z_n\}$. It is easy to see that $(z, \delta_1)$ is an extremal pair for $P$. Now, referrring to Theorem 2.2, we will show that the operator $E_P = \frac{1}{4}(w \otimes \delta_0) + \frac{3}{4}(z \otimes \delta_1)$ takes $U_1$ into $U$. Indeed, $U_1 = [u]$, where $u = \frac{1}{2}(\delta_0 + \delta_1) - \frac{1}{4}(\delta_0' + \delta_1')$, whence $\langle w, u \rangle = \frac{3}{2}$ and $\langle z, u \rangle = \frac{1}{2}$ leads to $E_P(u) = \frac{3}{8}(\delta_0 + \delta_1) = \frac{3}{4}u_1 \in U$. We conclude that $P$ is minimal by Theorem 2.2. ∎

*Note* 4.1. The above result can be interpreted as an application of Theorem 2.1. Indeed, with $S$ finite dimensional and $U_2=[\delta'_0, \delta'_1-\delta'_0]$, it follows that $S^*$, as defined above, is strictly dual-proper with respect to $U_2$.

DEFINITION 4.1. $x\in C[0,1]$ will be called *exact* provided $\int_0^1 x(t)\,dt=\frac{1}{2}(x(0)+x(1))$ (i.e.,the "trapezoidal rule" for approximating the integral of $x$ on $[0,1]$ is exact). $P$: $C^r[0,1]\to C^r[0,1]$ will be said to *preserve exactness* provided $(Px)^{(r)}$ is exact whenever $x^{(r)}$ is exact.

*Note* 4.2. For $u\in X^*$, the shape one generated by $\{u, -u\}$ determines a subspace of elements in $X$ with shape; i.e., $S$ is the nullspace of $S^*$. Exactness in $C^r[0,1]$, as defined above, can be interpreted as a shape generated by $\langle x, u\rangle=\int_0^1 x^{(r)}(t)\,dt-\frac{1}{2}(x^{(r)}(0)+x^{(r)}(1))$.

COROLLARY 4.1. *The projection in* (9) *is in fact minimal among the set of all exactness-preserving projections from* $C^1[0,1]$ *onto* 3-*dimensional subspaces interpolating the derivatives at* 0 *and* 1 *and preserving the average of the values at* 0 *and* 1.

*Proof.* First note that $x(t)=\cos\pi t\in U_\perp$ but $x\notin S=\ker(u)$ (see Note 4.2). Thus $U\cap S^*=\{0\}$ and this is the $k=0$ case of Lemma 1.3. Thus for any exactness preserving $P=\vec{u}\otimes\vec{v}$, with $\vec{v}=(v_1, ..., v_n)\in X^n$, we must have $v_i\in(S^*)^\perp$ for all $i$; note that this implies that all $v_i$ must have shape. Now, for $P$ in (9), we have

$$\int_0^1 (Px)'\,dt=Px(1)-Px(0)=\tfrac{1}{2}(x'(0)+x'(1))=\tfrac{1}{2}((Px'(0)+(Px)'(1)),$$

i.e., $(Px)'$ is always exact, whence $P$ automatically preserves shape (shape in this context is exactness of the first derivative). Furthermore, let $\phi_i\in C[0,1]$ such that $\tilde{P}=u_1\otimes\phi_1+u_2\otimes\phi_2+u_3\otimes\phi_3$ is a projection, where $u_1=\frac{1}{2}(\delta_0+\delta_1)-\frac{1}{4}(\delta'_0+\delta'_1)$, $u_2=\delta'_0$ and $u_3=\delta'_1$. As noted above, each $\phi_i$ must have shape. Now the orthogonality conditions lead to $\phi'_1(0)=\phi'_1(1)=0$ and $\frac{1}{2}(\phi_1(0)+\phi_1(1))=1$, while $\phi'_2(1)=0$, $\phi'_2(0)=1$, $\phi'_3(1)=1$, $\phi'_3(0)=0$, and $\frac{1}{2}(\phi_2(0)+\phi_2(1))-\frac{1}{4}=0$, $\frac{1}{2}(\phi_3(0)+\phi_3(1))-\frac{1}{4}=0$. Using these conditions together, with the fact that each $\phi_i$ has shape, we find $\phi_1(0)=1$ and $\phi_2(0)=\phi_3(0)=0$. Now since

$$\tilde{P}(x)=\phi_1(t)\,\frac{(x(0)+x(1))}{2}+x'(0)\left(\phi_2(t)-\frac{1}{4}\,\phi_1(t)\right)$$
$$+x'(1)\left(\phi_3(t)-\frac{1}{4}\,\phi_1(t)\right),$$

we have

$$\|\tilde{P}\| \geqslant \sup_{x \in \Sigma(X)} |\tilde{P}x(0)| \geqslant |\phi_1(0)| + |\phi_2(0) - \tfrac{1}{4}\phi_1(0)| + |\phi_3(0) - \tfrac{1}{4}\phi_1(0)| = 3/2.$$

We conclude that the projection (of norm 3/2) in (9) is in fact minimal among all exactness-preserving projections with kernel $U_\perp$ and the conclusion follows. ∎

Corollary 4.2. *The projection in* (9) *is in fact minimal among the set of all monotonicity-preserving projections preserving exactness from* $C^1[0, 1]$ *onto* 3*-dimensional subspaces interpolating the derivatives at* 0 *and* 1 *and preserving the average of the values at* 0 *and* 1.

*Proof.* For $P$ in (9), we have $(Px)'(t) = x'(0)(1-t) + x'(1)\,t$, whence $P$ preserves monotonicity. ∎

The above results extend to all $n \geqslant 2$ as follows.

Theorem 4.2. *Let* $(X, \|\cdot\|) = C^{n-1}[0, 1], \|\cdot\|)$, *where* $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, ..., \|x^{(n-1)}\|_\infty\}$. *Let* $U = [u_1, u_2, ..., u_n]$, *where* $(u_1, ..., u_n) = (\tfrac{1}{2}(\delta_0 + \delta_1), (\delta_1^k - \delta_0^k)_{k=0}^{n-3}, \delta_0^{n-1}, \delta_1^{n-1})$, *let* $A = I$ *and let* $S^*$ *be a shape cone such that* $S =$ *cone*$(t^{n-1}, t^n)$ (*the cone generated by the functions* $t^{n-1}$ *and* $t^n$). *I.e.*, $\mathscr{A}_{S^*}^U$ *is the set of all projections from* $C^{n-1}[0, 1]$ *onto* $n+1$*-dimensional subspaces containing* $t^{n-1}$ *and* $t^n$, *interpolating the* $(n-1)$*-st derivatives at* 0 *and* 1, *interpolating the differences of* $k$*th derivatives at* 0 *and* 1, $k = 0, ..., n-3$, *and preserving the average of the values at* 0 *and* 1. *Then there is an operator of minimal norm* (3/2 *if* $n \geqslant 2$ *and* 1 *if* $n = 1$) *and with range space* $\Pi_n$, *the subspace of all* $n$*th degree algebraic polynomials. This projection has then the explicit form*

$$P_n x(t) := (x(0) + x(1))/2 + \sum_{k=0}^{n-3} (x^{(k)}(1) - x^{(k)}(0))\, e_k(t)$$
$$+ x^{(n-1)}(1)\, d_{n-2}(t) + (-1)^{n+1}\, x^{(n-1)}(0)\, d_{n-2}(1-t), \quad (10)$$

*where*

$$e_k(t) := Te_{k-1}(t), \quad k = 1, 2, ..., \qquad and \qquad e_0(t) := t - 1/2 =: d_{-1}(t)$$

*and*

$$d_k(t) := Td_{k-1}(t), \quad k = 1, 2, ..., \qquad and \qquad d_0(t) := t^2/2 - 1/4,$$

*where*

$$Tx(t) := \left[ \int_0^t - t \int_0^1 \right] x(s)\, ds.$$

(*Note that* $T$ *provides the integral* $Ix(t)$ *of* $x$ *adjusted by a constant* $(c)$ *of* $t$ *so that* $Ix(t) - ct$ *vanishes at* 0 *and* 1.)

*Proof.* The proof follows by a fairly straightforward generalization of the proof of Theorem 4.1. (See [3] for an analogous discussion.) But, in order to illustrate how to find the operators, i.e., how to construct them geometrically via Theorem 3.2, consider the case $n=3$ as an illustration. We obtain

$$P_3 x(t) := x(1)\, t + x(0)(1-t) + x''(1)(1/6t^3 - 1/6t)$$
$$+ x''(0)(-1/6t^3 + 1/2t^2 - 1/3t)$$

as follows. First $U = [\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4]$, where $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4) = (\frac{1}{2}(\delta_0 + \delta_1), \delta_1 - \delta_0, \delta_0'', \delta_1'')$, and $S$ is generated by $t^2$ and $t^3$. Then $U_2 = [\delta_0'', \delta_1'']$ and $U_1 = U \cap S^{\perp} = [u_1, u_2]$, where $(u_1, u_2) = (\delta_0, \delta_1 - \frac{1}{6}(\delta_1'' + 2\delta_0''))$. Then $(S_4)_2 = \{(b_1, b_2) : \|b_1 u_1 + b_2 u_2\|_{X^*} = 1\}$. But $\|b_1 \delta_0 + b_2(\delta_1 - \frac{1}{6}(\delta_1'' + 2\delta_0''))\|_{X^*} = |b_1 + \frac{1}{2} b_2| = 1$ yields a diamond-shaped sphere with corners at $(0, 2)$ and $(1, 0)$. From this it follows from Theorem 3.2 where $y = \delta_t'$ that, $\langle (\vec{v})_2, y \rangle = (\vec{v})_2'(t) = \lambda(0, 2)$, for some constant $\lambda$. We conclude $(\vec{v})_2(t) = (c_1, c_2, t)$, for some constants $c_1, c_2$. ∎

COROLLARY 4.3. *The projection in* (10) *is in fact minimal among the set of all exactness-preserving projections from* $C^{n-1}[0, 1]$ *onto* $n+1$*-dimensional subspaces interpolating the* $(n+1)$*st derivatives at* 0 *and* 1, *interpolating the differences of* $k$*th derivatives at* 0 *and* 1, $k = 0, ..., n-3$, *and preserving the average of the values at* 0 *and* 1.

*Proof.* The proof follows by a direct generalization of the proof of Corollary 4.1. ∎

COROLLARY 4.4. *The projection in* (10) *is in fact minimal among the set of all* $(n-1)$*-convexity-preserving projections preserving exactness from* $C^{n-1}[0, 1]$ *onto* $n+1$*-dimensional subspaces interpolating the* $(n-1)$*st derivatives at* 0 *and* 1, *interpolating the differences of* $k$*th derivatives at* 0 *and* 1, $k = 0, ..., n-3$, *and preserving the average of the values at* 0 *and* 1.

*Proof.* The proof follows by a direct generalization of the proof of Corrolary 4.2. ∎

THEOREM 4.3. *Let* $(X, \|\cdot\|) = (C^{n-1}[0,1], \|\cdot\|)$, *where* $\|x\| = \max\{|x(0)|, |x'(0)|, ..., |x^{(n-2)}(0)|, \|x^{(n-1)}\|_\infty\}$. *Let* $U = [u_1, u_2, ..., u_n]$, *where* $(u_k = \delta_0^k$, $k = 0, ..., n$, *let* $A = I$, *and let* $S^* = \{\delta_t^{(n-1)}\}_{t \in [0,1]}$ (*thus* $S$ *is the cone of all* $(n- +1)$-*convex functions*). *I.e.*, $\mathscr{A}_{S^*}^{U}$ *is the set of all* $(n-1)$-*convexity-preserving projections from* $C^{n-1}[0,1]$ *onto* $n+1$-*dimensional subspaces interpolating the first* $(n-1)$ *derivatives at* $0$ *and the* $(n-1)$-*th derivative at* $1$. *Then there is an operator of minimal norm* $(1)$ *with range space* $\Pi_n$, *the subspace of all* $n$*th degree algebraic polynomials. This projection has then the explicit form*

$$P = \sum_{k=0}^{n-1} \frac{\delta_0^k}{k!} \otimes t^k + \frac{\delta_1^{n-1} - \delta_0^{n-1}}{(n-1)!} \otimes t^n.$$

*Proof.* It is immediate that $P$ is a projection with norm 1 with respect to the given norm and that it preserves $(n-1)$-convexity $((Px)^{(n-1)}(t) = x^{(n-1)}(0)\, t + x^{(n-1)}(1)(1-t))$. ∎

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